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ON A BOUNDARY VALUE PROBLEM OF NONLINEAR MAGNETOELASTICITY

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K. Sh. KHODZHAEV and I. Z. SHTILERMAN
(Leningrad)

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Equilibrium modes of a ferromagnetic string near a plane magnet and subjected to a uniformly distributed load are determined. The case when the initial spacing between the string and the magnet is commensurate with the displacements is examined. It is hence necessary to take account of the field dependence on the displacements despite the fact that these latter are small. Assumptions [1] are made permitting reduction of the problem to a nonlinear boundary value problem for just the displacements. The same boundary value problem is obtained upon determining the equilibrium in the field of a curved magnet. A number of solutions is found, the dependence of the solutions on the parameters is analyzed, and the stability of the equilibrium is investigated.

1. Nonlinear boundary value problem. Case of a load directed toward the magnet. Let us consider a slender, stretched ferromagnetic string attracted by an electromagnet and loaded by a uniformly distributed load q. We consider the magnetic permeability of the string, the magnet, and the magnetic circuit to be infinite, the lines of induction to be closed in conformity with Fig. 1, and enclosed by the same total current I. Let b be the width, and l the length of the string, Δ the spacing between the undeformed string and the magnet, and $u(x)$ the displacements of points of the string. It is assumed that $\Delta \ll b \ll l$, and u is commensurate with Δ. Keeping the lowest term in u/b in the expression for the ponderomotive force, we arrive at the boundary value problem [1]

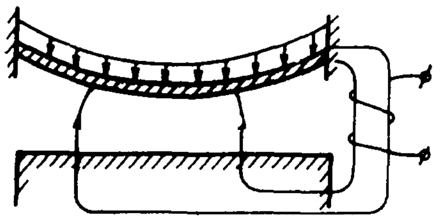

Fig. 1

$$v^{\cdot\cdot} + \frac{1}{(1-v)^2} + \gamma = 0, \qquad v(0) = v(\omega) = 0, \qquad v^{\cdot} = \frac{dv}{d\tau}$$

$$v = \frac{u}{\Delta}, \quad \tau = \frac{\omega}{l}x, \quad \omega^2 = \frac{\mu_0 b l^2 I^2}{2T\Delta^3}, \quad \gamma = \frac{2g\Delta^2}{\mu_0 b I^2} \tag{1.1}$$

Here v, τ are the dimensionless displacement and a coordinate, T is the tension, and μ_0 is the magnetic permeability of the medium. The same equation describes the equilibrium of a string bent by a magnet in the shape of a parabolic cylinder in the absence of an external load [1]. The solution of the boundary value problem (1.1) depends essentially on the sign of γ. Let $\gamma > 0$ (load directed toward the magnet). The equation in (1.1) hence has no singularities, and its first integral is

$$w^2 = 2(v_m - v)\,[(1 - v_m)^{-1}(1 - v)^{-1} + \gamma], \quad w = v^{\cdot} \tag{1.2}$$

Here v_m is the value of v upon intersection of the phase trajectory with the Ov-axis.

That one of the segments AB of the phase trajectories (Fig. 2a) which the describing point traverses during the "time" $\tau = \omega$ corresponds to the solution of (1.1). Since the lines AB are symmetric with respect to Ov, then the shape of the string is also symmetric relative to the axis passing through its center. The quantity v_m equals the maximum dimensionless displacement reached at the center of the string. Integrating (1.2), taking account of the condition $v(0) = 0$, we obtain

$$\tau(v) = \int_0^v \{2(v_m - z)\,[(1 - v_m)^{-1}(1 - z)^{-1} + \gamma]\}^{-1/2}\,dz, \quad 0 \leqslant \tau \leqslant \omega/2 \tag{1.3}$$

The shape of the string is determined by the relation (1.3) to the accuracy of the quantity v_m, which is found from the condition $v_m = v(\omega/2)$. The dependence $v_m = v_m(\omega)$ is thereby determined which describes the curve called ordinarily the equilibrium curve

$$\omega = 2\tau(v_m) \tag{1.4}$$

The function $\omega(v_m)$ is expressed in terms of elliptic integrals of the first and second kinds. But its inversion by such means is difficult, hence, let us start from expressions with quadratures. Evidently $\omega(0) = \omega(1) = 0$ for all $\gamma > 0$. We have

$$\frac{\partial\omega}{\partial v_m} = \frac{1}{\sqrt{2}}\int_0^1 \frac{dz}{\sqrt{1-z}}\,[g_1(z, v_m) - g_2(z, v_m)\,g_3(z, v_m)]$$

$$g_1 = \frac{1}{\sqrt{v_m}}[(1 - v_m)^{-1}(1 - v_m z)^{-1} + \gamma]^{-1/2}, \qquad g_3 = \frac{1 + z - 2v_m z}{[(1 - v_m)(1 - v_m z)]^{1/2}}$$

$$g_2 = \sqrt{v_m}\,[1 + \gamma(1 - v_m)(1 - v_m z)]^{-2} \tag{1.5}$$

The functions g_1, g_2 and g_3 are nonnegative, and g_2, g_3 increase monotonously, while g_1 decreases monotonously as v_m grows for $0 \leqslant z \leqslant 1$. Therefore, the integrand in (1.5) decreases monotonously as v_m grows. Hence, the derivative $\partial\omega/\partial v_m$ has not more than one zero for $0 < v_m < 1$. But since $\partial\omega/\partial v_m \to \infty$ as $v_m \to 0$ and $\partial\omega/\partial v_m \to -\infty$ as $v_m \to 1$, then there exists a value v_m for which $\partial\omega/\partial v_m = 0$. Therefore, in this case the function $\omega(v_m)$ has a single maximum $\omega = \omega_l$ (Figs. 3, 4). Correspondingly, for $\omega = \omega_l$ there exist two equilibrium modes for the same values of the parameters, one mode for $\omega > \omega_l$ while equilibrium is impossible for $\omega < \omega_l$.

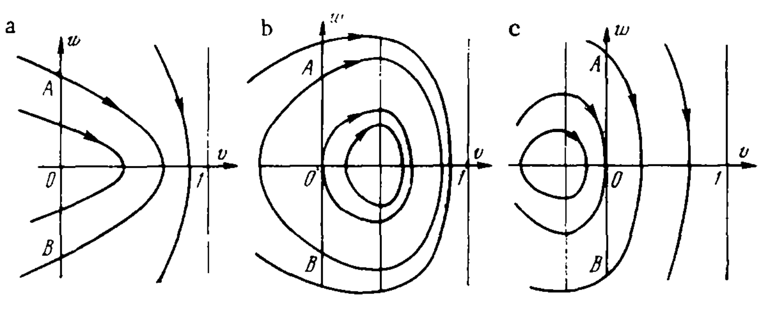

Fig. 2

2. Load directed away from the magnet. For $\gamma < 0$ Eq. (1.1) has the singularity $v = 1 - (-\gamma)^{-1/2}$, $w = 0$ in the Ovw, $v > 1$. The first integral of this equation

$$w^2 = 2\gamma\,(v_{m1} - v)\,(v_{m2} - v)\,(1 - v)^{-1}$$
$$(1 - v_{m1})\,(1 - v_{m2}) = -1 / \gamma \qquad (2.1)$$

determines closed phase trajectories intersecting Ov for $v = v_{m1}$ and $v = v_{m2}$. Hence, the singularity is a center at $\gamma < -1$ in the interval $0 < v < 1$, at $\gamma > -1$ in the interval $-\infty < v < 0$ (Fig. 2b, c).

Two groups of phase trajectory segments in addition to the others: segments of the form AB and BA (Fig. 2b, c), can correspond to the solution of the problem (1.1) in this case. The segments AB determine the equilibrium modes with positive displacements (see Sect. 1), and the segments BA those with negative displacements. These modes are symmetric and the maximum displacement in absolute value, equal to $v_{m1} > 0$ for "positive" and $v_{m2} < 0$ for "negative" modes, is reached at the center of the string. Moreover, equilibrium modes corresponding to the transitions ABA, $ABAB$ etc., are possible. It is necessary to select those among all these segments, which are traversed during the time $\tau = \omega$. The differences in the location of the center and the cases $\gamma < -1$ and $\gamma > -1$ influence the nature of the solution substantially.

Let $\gamma < -1$. Let us first examine positive one-extremal modes. The relationships (1.3), (1.4) are retained for them, but the limits of the possible variations in v_m in this case will be $v_p < v_m < 1$, where $v_p = 1 + 1/\gamma$. It is necessary that the integrand in (1.3) be real. Let us introduce the variable $t = zv_p / v_m$ in (1.3). We arrive at an integral with limits 0, v_p in (1.4) in which the integrand will be a monotonously deceasing function of v_m. Hence, as v_m increases from $v_m = v_p$ to $v_m = 1$, the function $\omega\,(v_m)$ decreases monotonously to zero. From (1.5) we have $\partial\omega / \partial v_m \to -\infty$ as $v_m \to 1$ and $v_m \to v_p$. Therefore, positive single-extremal modes exist in this case in the domain $0 \leqslant \omega \leqslant \omega_p\,(\gamma)$, where $v_m\,(\omega_p) = v_p$. such a mode is unique for given ω (Fig. 4).

Let us consider negative single-extremal modes. The dependence $\omega\,(v_m)$ for them is

$$\frac{\omega}{\sqrt{2}} = \int\limits_0^1 \left[\frac{|v_{m2}|\,(1 + |v_{m2}|)\,(1 + |v_{m2}|\,z)}{(1 - z)\,[-1 - \gamma\,(1 + |v_{m2}|)\,(1 + |v_{m2}|\,z)]}\right]^{1/2} dz \qquad (2.2)$$

The derivative of the integrand in (2.2) with respect to $|v_{m2}|$ is positive for any $|v_{m2}|$. Therefore $\omega\,(v_m)$ exists for all $|v_{m2}|$ and grows monotonously as $|v_{m2}|$ grows. Hence, for given ω a negative single-extremal mode exists and is unique for all $\omega \geqslant 0$ and

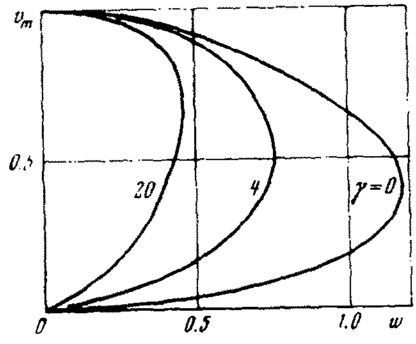

Fig. 3

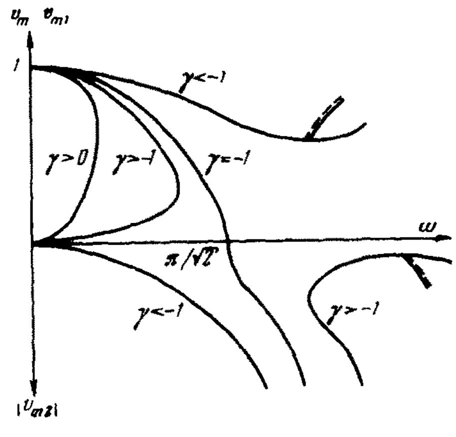

Fig. 4

$\omega(0) = 0$, $\partial\omega / \partial v_m \to -\infty$ for $|v_{m2}| \to 0$ (Fig. 4).

Three branches of multiextremal modes branch off from the branches of the positive single-extremal modes for $\omega = \omega_p$. These two modes are nonsymmetric, correspond to the paths ABA and BAB of the describing point (Fig. 2 b) and are obtained one from the other by a mirror reflection in the plane passing through the center of the string perpendicular to the Ox-axis. The third mode is symmetric and corresponds to the path $BABA$.

All these modes can be found as follows. Let us consider a positive single-extremal mode for some $\omega = \omega_1$, to which $v_{m1}(\omega_1)$ corresponds. Let us find a "conjugate" value v_{m2} from (2.1) and determine ω_2 from the relationship $v_{m2} = v_{m2}(\omega_2)$ for negative single-extremal modes. A two-extremal mode of type ABA corresponds to the value $\omega = \omega_1 + \omega_2$. A string having such a mode divides into two sections: on the first the mode coincides with a single-extremal mode for $\omega = \omega_1$, and on the second for $\omega = \omega_2$. Modes of the type $ABABA$ generated for $\omega = 2\omega_p(\gamma)$, etc., are determined analogously.

Let $\gamma > -1$. We consider positive single-extremal modes. They are defined by the relationships (1.3), (1.4), $0 \leqslant v_m \leqslant 1$ as in the case $\gamma > 0$. Let us show that the dependence $\omega = \omega(v_m)$ has just one maximum. As before, $\partial\omega / \partial v_m \to \infty$ as $v_m \to 0$ and $\partial\omega / \partial v_m \to -\infty$ as $v_m \to 1$, i.e. $\omega(v_m)$ has at least one maximum. We assume that one maximum is split into several maxima and minima for $\gamma = \gamma_*$. For $\gamma = \gamma_*$ let the maximum be reached for $v_m = v_{m*}$. Then we should have

$$\partial\omega/\partial v_m = 0, \qquad \partial^2\omega/\partial v_m \partial\gamma = 0 \tag{2.3}$$

for $v_m = v_{m*}$, $\gamma = \gamma_*$. It can be shown that the equalities (2.3) are inconsistent. Hence, it follows that the generation of new extrema is impossible as γ varies in $(-1, 0)$ and the curves $\omega(v_m)$ have just one maximum (Fig. 4).

In this case negative single-extremal equilibrium modes are also possible. The dependence $\omega(|v_{m2}|)$ for them is defined as before by the relationship (2.2), where however $|v_{m2}| \geqslant v_p = |1 + 1/\gamma|$. Let us find the derivative $\partial\omega / \partial |v_{m2}|$, which has the form (1.5), but v_m is replaced by $-|v_{m2}|$. For $|v_{m2}| = v_p$ the function g_3 in (1.5) goes over into $g_2 = z^{-1/2} v_p^{-1}$ and the corresponding integral diverges. On the other hand, for $|v_{m2}| \to \infty$ the function $g_1(z, |v_{m2}|)$ decreases as $|v_{m2}|^{-1/2}$, and the product $g_2 g_3$ as $|v_{m2}|^{-3/2}$. Hence, for some $|v_{m2}|$ the integrand in the expression for the derivative will be positive. Therefore, the branch of the negative modes has a limit point (Fig. 4).

The branch $\omega_p = \omega(v_p)$ of the single-extremal modes emanating from the limit point toward smaller $|v_{m2}|$ splits into three branches at the point (ω_p, v_p), the branch of the three-extremal symmetric modes $ABAB$ and two branches of the two-extremal nonsymmetric modes ABA and BAB. These modes can be found by the above-mentioned addition method.

There remains to consider the case $\gamma = -1$. In this case, there is a solution $v \equiv 0$ corresponding to an undeformed string for any ω. For $\gamma = -1$, we obtain from (1.3), (1.4)

$$\frac{\omega}{2} = \sqrt{1 - v_m} \int_0^1 \sqrt{\frac{1 - v_m z}{1 - z}} \frac{dz}{\sqrt{1 + (1 - v_m) z}} \tag{2.4}$$

For $v_m > 0$ the relationship (2.4) describes a branch of single-extremal modes into which the branch of single-extremal positive modes corresponding to the case $\gamma > -1$ and located between the limit point and the point $v_m = 1$ goes for $\gamma \to -1 + 0$

Another ("lower") branch of these modes goes into segments of the ω-axis between the point $\omega = 0$ and the point of intersection of the ω-axis and the lines (2.4). Setting $v_m = 0$ into (2.4), we obtain $\omega = \pi / \sqrt{2}$. The limit point of the branch of positive single-extremal modes for $\gamma \to -1 + 0$ arrives at the point $\omega = \pi / \sqrt{2}$, $v_m = 0$ hence the tangent to the curve $v_m(\omega)$ is vertical at this point (Fig. 4).

The limit point of branches of the negative single-extremal modes also arrive at the point $\omega = \pi / \sqrt{2}$, $v_m = 0$ as $\gamma \to -1 - 0$. The segments of this branch which are located between the limit point and the branch point go over into the segments $\pi / \sqrt{2} < \omega \leqslant \pi \sqrt{2}$ of the ω-axis and its infinite part goes over into an infinite branch of single-extremal modes which are described by the same expression (2.4) if v_m is replaced by $-|v_{m2}|$. Moreover, an infinite number of branches of multiextremal modes theoretically branch off from the solution $v \equiv 0$ for values of ω which are multiples of $\pi / \sqrt{2}$.

3. Stability of the equilibrium. Equation (1.1) can be derived from the variational principle

$$\delta V = 0, \qquad V = \int_0^\omega \left(\frac{1}{2} v'^2 - \frac{1}{1 - v} - \gamma v \right) d\tau \tag{3.1}$$

where functions $v(\tau)$ such that $v(0) = v(\omega) = 0$, $v'(\tau) \in L_2(0, \omega)$ and $v(\tau) < 1$ are admitted for comparison. We consider an equilibrium mode stable if it renders the functional V a local minimum in the mentioned class of functions. Let us investigate the stability of the undeformed state for $\gamma = -1$. The second variation of the functional (3.1) in the solution $v \equiv 0$ has the form

$$\delta^2 V = \int_0^\omega \left(\frac{1}{2} \zeta'^2 - \zeta^2 \right) d\tau \tag{3.2}$$

where $\zeta(0) = \zeta(\omega) = 0$ and $\zeta'(\tau) \in L_2(0, \omega)$. For such $\zeta(\tau)$ we have

$$\int_0^\omega \zeta'^2 \, d\tau \geqslant \frac{\pi^2}{\omega^2} \int_0^\omega \zeta^2 \, d\tau \tag{3.3}$$

where such $\zeta(\tau)$ exist for larger coefficients in front of the second integral that the opposite inequality is satisfied. Hence, $\delta^2 V$ will be greater than zero for $\omega < \pi / \sqrt{2}$, i.e. the undeformed state is stable up to the bifurcation point $\omega = \pi / \sqrt{2}$. At the bifurcation point it becomes unstable to the first degree of instability, and the stability goes over to the branch of the negative single-extremal modes. At the next singular point $\omega = \pi \sqrt{2}$ the undeformed state becomes unstable to the second degree of instability, and the departing branches will be of the first degree of instability, etc., as in the longitudinal bending problem.

Let us consider positive single-extremal modes for $\gamma > -1$. We take $v(\omega, \gamma)$ on the branch with smaller v_m. For $\omega = \text{const}$ and $\gamma \to -1$ this mode goes continuously over into a stable undeformed state. The stability does not vary in such a passage, hence this mode is stable. It follows from the shape of the equilibrium curves $v_m(\omega)$ for $\gamma = \text{const}$ that stability vanishes at the limit point, and the branch with the greater v_m is unstable. By the same means we find that the branch of the negative single-extremal modes going to infinity from the limit point is stable. Negative single-extremal modes are also stable for $\gamma < -1$. All the remaining modes are unstable.

It is remarkable that for $0 > \gamma > -1$ there exist two nonintersecting stable branches, where the equilibrium modes have inflection points of the negative branch; this latter is due to the interchange of the growth and diminution of $\omega(\tau)$ (Fig. 2 c). For comparison, let us show that in a similar problem examined in [2] and in a number of problems on Euler elastica, the modes with an inflection are known to be unstable.

The displacements are determined as the sum of the function $v(\tau)$ found earlier and a second degree polynomial in τ in the problem of equilibrium of a string without an external load under the effect of a curved magnet. Hence, only positive forms without inflection points are obtained, as it should be also in the case of a load which does not change sign for any displacements.

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ON GENERAL RELATIONSHIPS OF THE THEORY OF IDEAL PLASTICITY AND THE STATICS OF A FRIABLE MEDIUM

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D. D. IVLEV
(Moscow)
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General relationships of the theory of ideal plasticity and the statics of a friable medium for Tresca plasticity condition and its extensions, on the basis of determining the dissipation function, are considered. The work is related to the investigations in [1, 2].

1. Under the Tresca plasticity condition, the dissipation function is

$$D = 2k \mid \varepsilon_i \mid_{\max}, \qquad k = \text{const} \tag{1.1}$$

where $\varepsilon_{i\,\max}$ is the maximum principal strain rate component. For definiteness, we later assume $\varepsilon_i = \varepsilon_3$; we shall consider the material incompressible. Let us write the initial functional to determine the associated loading law as

$$D = 2k\, \varepsilon_3 (\varepsilon_{ij}) + \mu (\varepsilon_x + \varepsilon_y + \varepsilon_z) \tag{1.2}$$

where ε_{ij} are the components of the strain rate tensor, μ is a Lagrange multiplier. It is necessary to know the expression $\varepsilon_3 = \varepsilon_3(\varepsilon_{ij})$. Let n_i denote the direction cosines of the third principal direction in a Cartesian coordinate system x_i. Then $n_i \varepsilon_3 = \varepsilon_{ij} n_j$. Hence, the known formula follows

$$\varepsilon_3 = \varepsilon_{ij}\, n_i n_j \tag{1.3}$$

Using (1.3), it is necessary to take into account that $n_i = n_i(\varepsilon_{ij})$ since the orientation of the principal directions change when the components of the strain rate tensor change. Taking account of (1.2), (1.3), in conformity with the associated loading law we obtain